Solar System test for the existence of gravitational waves

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Abstract

Starting from a static spherically symmetric solution of the Einstein's field equations in the second approximation in the perfect fluid scheme, in the exterior of the source, the corresponding metrics depend on a dimensionless parameter α . Taking the Sun for the source, the influence of α on the classical Solar System tests of the general relativity is estimated, the only α-dependent one being the Shapiro radar-echo delay experiment. Performing such a test in given conditions within a precision better than 10^{-2}, it is possible to obtain an experimental value for α . If $\alpha = 1$, the Einstein's equations in the weak field approximation take the D'Alembert form, which attests the existence of gravitational waves[1].

Keywords: general relativity - solar system tests - exact solutions

1 Introduction

For the metric theories of gravitation, the retardation of light signal passing near massive bodies provides a principal test (Shapiro, 1964-1989, Weinberg, 1972, Reasenberg et al., 1979). In the following, we shall perform a more accurate analysis of this effect, starting from its "anisotropy", i.e. the direction dependence of the light velocity in the gravitational field of the Sun - which distinguishes it from other classical Solar System tests. The origin of the anisotropy is to be found in the manner of gauging the gravitational potentials, the effect being evidenced by means of a certain dimensionless parameter α. In the second section, a statically symmetric solution of the Einstein's field equations in the second approximation for a spherical mass distribution is given and the equations of motion for both massive bodies and photons are derived. In the third section, the influence of this metrics on the Solar System tests of General Relativity is estimated. The fourth section concerns with the connection between the Shapiro test and the propagation of gravitational waves.

2 The static field of the Sun and the equations of motion

2.1 The static metrics

We start from the standard static spherically symmetric metrics (Zeldovich, Novikov, 1971):

$$(dS)^2 = e^{\nu(r)}(cdt)^2 - \left[e^{\lambda(r)}(dr)^2 + e^{\sigma(r)}r^2 d\Omega\right] \tag{1}$$

where $\nu(r)$, $\lambda(r)$ and $\sigma(r)$ are arbitrary functions to be established from physical arguments. By asking this diagonal metric to fulfill the Einstein's field equations:

$$E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu} \qquad \mu,\nu = 0,1,2,3 \tag{2}$$

in the perfect fluid scheme (Misner, Thorne, Wheeler, 1973), we come to the following expressions for the nonzero components of $E_{\mu\nu}$ and $T_{\mu\nu}$:

$$\begin{gathered}
E_{00} = e^{\nu-\lambda}\left[(\Delta\sigma + \tfrac{3}{4}\sigma'^2) + \tfrac{1}{r}(\sigma' - \lambda') - \tfrac{1}{2}\lambda'\sigma' + \tfrac{1}{r^2}\right] - \tfrac{1}{r^2}e^{\nu-\sigma} \\
E_{11} = -\left[\tfrac{1}{4}\sigma'^2 + \tfrac{1}{2}\nu'\sigma' + \tfrac{1}{r}(\sigma' + \nu') + \tfrac{1}{r^2}\right] + \tfrac{1}{r^2}e^{\lambda-\sigma} \\
E_{22} = r^2 e^{\sigma-\lambda}\left[-\tfrac{1}{2}(\Delta\sigma + \tfrac{1}{2}\sigma'^2) - \tfrac{1}{2}(\Delta\nu + \tfrac{1}{2}\nu'^2) + \tfrac{1}{2r}(\nu' + \lambda') + \tfrac{1}{4}\sigma'(\lambda' - \nu') + \tfrac{1}{4}\nu'\lambda'\right] \\
E_{33} = (\sin^2\theta)\cdot E_{22} \\
T_{00} = e^{\nu}\left[\rho c^2 + \left(\rho\int_0^p \tfrac{dp}{\rho} - p\right)\right] \\
T_{11} = pe^{\lambda} \qquad T_{22} = r^2 pe^{\sigma} \qquad T_{33} = (\sin^2\theta)T_{22}
\end{gathered} \tag{3}$$

where ρ is the invariant rest mass density and p is the invariant pressure of the source fluid; we denoted by prime the derivative with respect to r . In the linear approximation, the field equations become:

$$\begin{gathered}
\Delta\sigma + \tfrac{1}{r}(\sigma' - \lambda') + \tfrac{1}{r^2}(\sigma - \lambda) = -\tfrac{8\pi G}{c^2}\rho \\
-\tfrac{1}{r}(\sigma' + \nu') - \tfrac{1}{r^2}(\sigma - \lambda) = 0 \\
\Delta\sigma + \Delta\nu - \tfrac{1}{r}(\nu' + \lambda') = 0
\end{gathered} \tag{4}$$

Assuming the source to be a sphere of radius R , we have:

$$M_0 = \frac{1}{c^2}\int_0^R \rho_E(r)\cdot 4\pi r^2 dr \qquad \Delta\Phi = -4\pi\frac{G}{c^2}\rho_E \tag{5}$$

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where ρ_E is the total rest energy density of the source, M_0 is the total rest mass and Φ the Newtonian potential. In the linear approximation, $\rho_E = \rho c^2$. In the second order approximation, the gravitational self energy as well as the elastic energy coming from the equilibrium pressure are included. The solution of 4 is:

$$\nu = -2\frac{\Phi}{c^2} \qquad \lambda = (r\sigma)' - 2\frac{r\Phi'}{c^2} \qquad \sigma - \text{arbitrary} \tag{6}$$

(see Tonnelat, 1965 about the arbitrariness of σ).

Outside the mass distribution, an accurate solution of the field equations $E_{\mu\nu} = 0$ is:

$$(dS)^2 = \left[1 - 2\frac{\mu}{f(r)}\right](cdt)^2 - \left\{\left[1 - 2\frac{\mu}{f(r)}\right]^{-1} f'^2(r)(dr)^2 + f^2(r)d\Omega\right\} \tag{7}$$

where, due to the asymptotic flatness, $f(r)$ is subjected to the constraint:

$$\lim_{r\to\infty} \frac{f(r)}{r} = 1 \tag{8}$$

The solution 7 may be checked by direct standard calculations. The integration constant μ is determined as:

$$\mu = \frac{GM_0}{c^2} \quad , \tag{9}$$

from the necessity of regaining Newtonian theory as a non-relativistic limit. From 8 we can write $f(r)$ as an expansion:

$$f(r) = r + \alpha\mu + O(\frac{\mu^2}{r}) \tag{10}$$

provided that r and μ are the only arguments of f with physical dimensions. Here α is a dimensionless parameter. Using 10 we get an explicit approximate form of the metrics 7:

$$(dS)^2 = \left[1 - 2\frac{\mu}{r} + 2\alpha\left(\frac{\mu}{r}\right)^2\right](cdt)^2 - \left[\left(1 + 2\frac{\mu}{r}\right)(dr)^2 + \left(1 + 2\alpha\frac{\mu}{r}\right)r^2 d\Omega\right] \tag{11}$$

A little discussion is here needed. Outside the source, derived from 1 and 11, we have $\sigma \approx 2\alpha\frac{\mu}{r}$. On the other side, the quantities ν, λ, σ cannot contain Φ'', due to difficulties concerning the continuity conditions on the surface of the spherical source. The only possibility in hand is therefore $\sigma \approx 2\alpha\Phi$, which leads to:

$$\begin{aligned}(dS)^2 \approx \left(1 - 2\tfrac{\Phi}{c^2} + 2\alpha\tfrac{\Phi^2}{c^4}\right)(cdt)^2 - \\ - \left\{\left[1 + 2\alpha\tfrac{\Phi}{c^2} - 2(1-\alpha)\tfrac{r\Phi'}{c^2}\right](dr)^2 + \left(1 + 2\alpha\tfrac{\Phi}{c^2}\right)r^2 d\Omega\right\}\end{aligned} \tag{12}$$

The going through the solution 12 ensures the compatibility between the inner and outer solutions within the given approximation. Actually, it was proved that the metrics 12 is a second order approximate solution of the field equations 2 provided that the rest energy density and the equilibrium pressure are:

$$\rho_E = \left(1 - \frac{GM_0}{c^2 R}\right)\rho c^2 + (\alpha - 1)(\Phi + 2r\Phi')\rho + 2p \qquad p = -\int_r^R \rho\Phi' dr \quad (13)$$

Details are given in Appendix A.

2.2 The equations of motion

The equations of motion are resulting from the geodetic variational principle, which is equivalent to the Euler-Lagrange variational principle:

$$\delta \int L dt = 0 \qquad L \equiv -m_0 c^2 K \qquad K \equiv \frac{dS}{cdt} = \left[A - \left(B\frac{\mathbf{v}^2}{c^2} + C\frac{(\mathbf{v}\cdot\mathbf{r})^2}{c^2 r^2}\right)\right]^{1/2} \quad (14)$$

$$\begin{gathered} \tfrac{d}{dt}\left(\tfrac{\partial L}{\partial \mathbf{v}}\right) - \tfrac{\partial L}{\partial \mathbf{r}} = 0 \qquad \tfrac{d\mathbf{p}}{dt} - \mathbf{F} = 0 \\ \mathbf{p} = \tfrac{\partial L}{\partial \mathbf{v}} = -\tfrac{1}{2}E\tfrac{1}{A}\tfrac{\partial}{\partial \mathbf{v}}(K^2) \qquad \mathbf{F} = \tfrac{\partial L}{\partial \mathbf{r}} = -\tfrac{1}{2}E\tfrac{1}{A}\tfrac{\partial}{\partial \mathbf{r}}(K^2) \end{gathered} \quad (15)$$

Here we used the energy integral:

$$E = \mathbf{v}\cdot\mathbf{p} - L = \frac{m_0 c^2}{K}A \quad (16)$$

in order to eliminate the quantity $m_0 K^{-1}$ in the expressions of the momentum $\mathbf{p}$ and the force $\mathbf{F}$, because $m_0 K^{-1}$ becomes undetermined, 0/0, in the case of the photon. From 14 and 15, we get:

$$\mathbf{p} = -\frac{1}{2}\frac{m_0 c^2}{K}\frac{\partial}{\partial \mathbf{v}}K^2 \qquad \mathbf{F} = -\frac{1}{2}\frac{m_0 c^2}{K}\frac{\partial}{\partial \mathbf{r}}(K^2) \quad (17)$$

As $d\mathbf{E}/dt = 0$, the equation of motion acquires a form which holds for both massive bodies and photons:

$$\frac{d}{dt}\left[\frac{1}{A}\frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}K^2\right)\right] - \frac{1}{A}\frac{\partial}{\partial \mathbf{r}}\left(\frac{1}{2}K^2\right) = 0 \quad (18)$$

Comparing dS/cdt from 14 to the homologous quantity from 11, we get:

$$A \approx 1 - 2\frac{\mu}{r} + 2\alpha\left(\frac{\mu}{r}\right)^2 \qquad B \approx 1 + 2\alpha\frac{\mu}{r} \qquad C \approx -2(\alpha - 1)\frac{\mu}{r} \quad (19)$$

By using series expansions relying on $\mu/r \ll 1$, we finally derived the acceleration experienced by a point-like body of rest mass m_0, $m_0 \ll M_0$, outside the source:

$$\mathbf{a} \approx -\frac{GM_0}{r^3}\left[1 - 2C_1\frac{\mu}{r} + 2C_2\boldsymbol{\beta}^2 - 2C_3\frac{(\boldsymbol{\beta}\cdot\mathbf{r})^2}{r^2}\right]\mathbf{r} + 2C_4\frac{GM_0}{r^3}(\boldsymbol{\beta}\cdot\mathbf{r})\boldsymbol{\beta} \quad (20)$$

where:

$$C_1 = \alpha{+}1 \qquad C_2 = 1{-}\frac{\alpha}{2} \qquad C_3 = -\frac{3}{2}(\alpha{-}1) \qquad C_4 = \alpha{+}1 \qquad \boldsymbol{\beta} = \frac{\mathbf{v}}{c} \quad (21)$$

3 The Solar System general relativistic tests

From the results of section II, performing usual computations (Weinberg, 1972), we derived formulas for estimating the general relativistic tests, within the Solar System, for an arbitrary gauge condition. The results are shown in the sequel.

The *perihelion advance* of Mercury per revolution $\delta\omega$ may be estimated by a standard procedure. The equation 20 is integrated in order to obtain the energy and angular momentum integrals:

$$\begin{array}{c} \left(\frac{1}{2}\mathbf{v}^2 - \frac{GM_0}{r}\right) + \frac{GM_0}{c^2 r}[(2C_4 - 2C_2 + \frac{4}{3}C_3)\mathbf{v}^2 + \frac{2}{3}C_3\frac{(\mathbf{v}\cdot\mathbf{r})^2}{r^2} \\ +(C_1 + 2C_2 - 2C_3 - 2C_4)]\frac{GM_0}{r} = \mathcal{E} \\ \left(1 + 2C_4\frac{GM_0}{c^2 r}\right)(\mathbf{v}\times\mathbf{r}) = \mathbf{A} \end{array} \qquad (22)$$

$$\mathcal{E} = 4\frac{E}{m_0} - \left(\frac{5}{2}c^2 + \frac{3}{2}\frac{E^2}{m_0^2 c^2}\right) \quad , \qquad \mathbf{A} = \frac{c^2}{E}\mathbf{M}$$

where $\frac{1}{2}\mathbf{A}$ is the constant areolar velocity, E the energy and $\mathbf{M}$ the angular momentum. Out of these equations, a relativistic Binet equation may be derived (Ionescu-Pallas, 1980):

$$\frac{d^2(1/r)}{d\theta^2} + k\cdot\left(\frac{1}{r}\right) = \frac{1}{p_1} + 2C_3\epsilon^2\frac{\mu}{p_0^2}\cos^2\theta \qquad (23)$$

where $k = 1{+}2\cdot\frac{\mu}{p_0}\cdot(C_1{-}2C_2{-}2C_4)$, $\mu = GM_0/c^2$, $p_0 = a(1{-}\epsilon^2)$, $p_1 = p_0{+}O(\mu)$. Then, a perturbational technique is used for reaching the trajectory formula:

$$\frac{1}{r} = \frac{1}{kp_1}\left[1 + \epsilon\cos\left(\theta\sqrt{k}\right)\right] + \frac{2}{3}C_3\epsilon^2\frac{\mu}{p_0^2}\left(1 + \sin^2\theta\right) \quad . \qquad (24)$$

After such calculations, $\delta\omega$ turns out to have the expression:

$$\delta\omega = (-C_1 + 2C_2 + 2C_4)\frac{2\pi GM_0}{c^2 a(1-\epsilon^2)} = \frac{6\pi GM_0}{c^2 a(1-\epsilon^2)} \qquad (25)$$

which is independent on α. Here M_0 is the rest mass of the Sun, a - semi major axis of the ecliptic of Mercury, ϵ - Mercury ecliptic eccentricity.

The *light deflection* angle $\delta\Psi$ is defined as:

$$\delta\Psi = -\frac{1}{c}\int_{-\infty}^{+\infty} a_x(t)dt = -\frac{1}{c}\left[v_x(+\infty) - v_x(-\infty)\right] \qquad (26)$$

where:

$$a_x = -\frac{GM_0}{r^3} R\left[(1+2C_2) - 2C_3\cos^2\theta\right] \qquad r = \left(c^2t^2 + R^2\right)^{\frac{1}{2}} \qquad \cos\theta = \frac{ct}{R} \tag{27}$$

Here, R is the Sun radius; a_x is obtained from 20 by replacing there $\mathbf{v} \sim c\mathbf{j}$ and $x = R$, $y = ct$. v_x is the projection of the photon velocity on the Ox axis (transverse to the trajectory) (Ionescu-Pallas, 1980). After standard calculations, we get:

$$\delta\Psi = \left(1 + 2C_2 - \frac{2}{3}C_3\right)\frac{2GM_0}{c^2R} = \frac{4GM_0}{c^2R} \quad , \tag{28}$$

that is $\delta\Psi$ is also independent on α .

Obviously, the *gravitational red shift* is independent on α , due to its connection only to the linear approximation of g_{00} .

However, the situation turns not to be the same with the *Shapiro retardation effect* (see for the following Shapiro, 1964, 1989, Weinberg, 1972, Misner, Thorne, Wheeler, 1973). We shall derive the formula for δT - the retardation of a radar signal in a to-and-fro Terra-Mercury journey, when Mercury is at the upper conjunction. The elementary duration spent by a photon is, from 11:

$$cdt \approx \left[1 + 2\frac{\mu}{r} + (\alpha - 1)R^2\frac{\mu}{r^3}\right]\frac{rdr}{\sqrt{r^2 - R^2}} + R\frac{d(\theta - \theta_0)}{dr}dr \tag{29}$$

The photon is travelling, let's say, almost along the Oy axis, passing at the nearest distance R from the center of the Sun. The first term in 29 leads to the retardation due to the gravitational refractive index n along the non-deviated trajectory $(x = R)$

$$n = 1 + 2\frac{\mu}{r} + (\alpha - 1)R^2\frac{\mu}{r^3} \tag{30}$$

The second term leads to the "overtaking" retardation. There are no problems to estimate the first effect. For the estimation of the second effect, we first notice that $\theta_0(r) = \arccos\frac{R}{r}$ is the equation of the line $x = R$ (the trajectory in the absence of the gravitational field). Thereafter, we make use of the trajectory equation:

$$\frac{1}{r} \approx \frac{\cos\theta}{R} + \frac{\mu}{R^2}\left[2 - (\alpha + 1)\cos\theta + (\alpha - 1)\cos^2\theta\right] \tag{31}$$

in order to get the formula:

$$(\theta - \theta_0) \approx \frac{\mu}{R}\frac{r}{\sqrt{r^2 - R^2}}\left[2 - (\alpha + 1)\frac{R}{r} + (\alpha - 1)\left(\frac{R}{r}\right)^2\right] \tag{32}$$

whence

$$R\frac{d}{dr}(\theta - \theta_0) = \frac{\mu R}{\sqrt{r^2 - R^2}}\left[\frac{\alpha + 1}{r + R} - (\alpha - 1)\frac{R}{r^2}\right] \tag{33}$$

The integration is straightforward. We obtain for the retardation δT the formula:

$$\delta T = \frac{4GM_0}{c^3}\left[\ln\left(\frac{4l_1 l_2}{R^2}e^{\alpha-1}\right)+2\right] = [220.5+19.7(\alpha+1)]\times 10^{-6}s \tag{34}$$

where l_1 is the major semi-axis of the ecliptic of Terra and l_2 is the major semi-axis of the ecliptic of Mercury. Performing a Shapiro-type experiment within an error of $\leq 10^{-2}$ it is possible to determine an experimental value of α.

4 The gauge parameter α and the gravitational waves propagation

We write the general metric tensor as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{35}$$

where $\eta_{\mu\nu} = 2\delta_{0\mu}\delta_{0\nu} - \delta_{\mu\nu}$ is the covariant Minkowski tensor. From 11 and 19, we have, in Cartesian coordinates:

$$h_{00} = (A-1) \qquad h_{jk} = -(B-1)\delta_{jk} - C\frac{x_j x_k}{r^2} \quad , \qquad j,k = 1,2,3 \tag{36}$$

Imposing the well-known Hilbert gauge condition (Zeldovich, Novikov, 1971, Tonnelat, 1965):

$$\left(h^{\lambda\sigma} - \frac{1}{2}\eta^{\lambda\sigma}h\right)_{,\lambda} = 0 \qquad \lambda,\sigma = 0,1,2,3 \qquad (\quad)_{,\lambda} \equiv \frac{\partial}{\partial x^\lambda}(\quad) \tag{37}$$

where $\eta^{\lambda\sigma} = 2\delta^{0\lambda}\delta^{0\sigma} - \delta^{\lambda\sigma}$ is the contravariant Minkowski tensor, with:

$$h = \eta^{\lambda\sigma}h_{\lambda\sigma} \approx 4\alpha\frac{\mu}{r} \tag{38}$$

we get from 36, 37 and 38, by replacing the coefficients A, B, C with their values from 19:

$$\alpha = 1 \tag{39}$$

i.e. the condition for α in order to satisfy the Hilbert gauge.

From the other point of view, Einstein's field equations 2, written in rectangular coordinates and in an inertia frame:

$$R_{\mu v} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \tag{40}$$

reduce, with $T_{\mu\nu}$ in the perfect fluid scheme, and within the linear approximation 35 ($|h_{\mu\nu}| \ll 1$), to:

$$\Box h_{\mu\nu} - \left[\left(h^\lambda_\mu - \frac{1}{2}\delta^\lambda_\mu h\right)_{,\lambda\nu} + \left(h^\lambda_\nu - \frac{1}{2}\delta^\lambda_\nu h\right)_{,\lambda\mu}\right] = -\frac{8\pi G}{c^2}\rho\left(u_\mu u_\nu - \frac{1}{2}\eta_{\mu\nu}\right) \tag{41}$$

where:

$$u_\mu = \eta_{\mu\lambda} u^\lambda \qquad u^\lambda = \frac{dx^\lambda}{dS_M} \qquad (dS_M)^2 = \eta_{\lambda\sigma} dx^\lambda dx^\sigma$$
$$\Box \equiv \eta^{\lambda\sigma} \frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^\sigma} = -\left(\triangle - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \tag{42}$$

For bringing 40 to the D'Alembert form, we need just the condition 37, i.e. $\alpha = 1$. The D'Alembert form unambiguously attests the existence of gravitational waves, in the framework of Einstein's General Relativity Theory.

5 Conclusions

We performed a more accurate analysis of the Shapiro time-delay Solar System test of general relativity. In order to perform the analysis, in the second section of the paper we studied a general spherically symmetric static metrics given by the Sun, in the second approximation. Starting from a more general solution of the Einstein's equations 7 and imposing the physical conditions of regaining the Newtonian limit and the continuity of the solution on the surface of the spherical mass source, we obtained an approximate form of the metrics in the exterior of the mass source 11, which contains, in the second approximation an undetermined dimensionless parameter α. By means of standard calculations performed using the metrics 11, we got a form of the equations of motion 18 for point masses, which holds for both massive bodies and photons, and we derived the acceleration 20 experienced by a point-like massive body or photon, both 18 and 20 holding for the movement in the static field of the Sun. In the third section, we estimated the influence of the undetermined gauge parameter α on the classical Solar System tests. The perihelion advance, the light deflection and the gravitational red shift are independent on α. However, the Shapiro effect depends on α. Performing the calculations for the retardation of the radar signal in a to-and-fro Terra-Mercury journey, with Mercury at the upper conjunction, we estimated a relative influence of α of the order $\sim 10^{-2}$ on the total signal retardation of time 34. Finally, in the fourth section, we established the connection of the parameter α to the propagation of the gravitational waves. From the Hilbert gauge condition 37, necessary for the Einstein's equations in order to take the D'Alembert form (which attests the existence of gravitational waves), we obtained the value $\alpha = 1$.

The short conclusions of our analysis are the following: *i)* it is possible to determine an experimental value for α by means of a Shapiro-type experiment, and *ii)* if $\alpha = 1$, the existence of general relativistic gravitational waves is proved.

Finally, a definite alternative - which may be solved experimentally through the agency of the Shapiro effect - is reached: either the light propagation in the gravitational field of the Sun is isotropic and the gravitational waves do exist, or the respective propagation is anisotropic and the gravitational waves

are precluded. It is to point out that our conclusions are conditioned by the hypothesis about the correctness of Einstein's General Relativity Theory.

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6 Appendix

All the physical quantities entering the Einstein's field equations of General Relativity Theory are expanded in power series of the constant $1/c^2$:

$$\nu = \frac{1}{c^2}\nu_0 + \frac{1}{c^4}\nu_1 + O\left(\frac{1}{c^6}\right) \tag{43}$$

$$\lambda = \frac{1}{c^2}\lambda_0 + \frac{1}{c^4}\lambda_1 + O\left(\frac{1}{c^6}\right) \tag{44}$$

$$\sigma = \frac{1}{c^2}\sigma_0 + \frac{1}{c^4}\sigma_1 + O\left(\frac{1}{c^6}\right) \tag{45}$$

$$\rho = \rho_M - \frac{1}{c^2}\rho_M\left(\frac{1}{2}\lambda_0 + \sigma_0\right) + O\left(\frac{1}{c^4}\right) \tag{46}$$

$$p = p_M + O\left(\frac{1}{c^2}\right) \tag{47}$$

(the label M stands for the Minkowskian nature of the respective physical quantity). Accordingly, the tensors $E_{\alpha\beta}$ and $T_{\alpha\beta}$ are themselves expanded in similar power series:

$$E_{\alpha\beta} = \frac{1}{c^2}E^{(0)}_{\alpha\beta} + \frac{1}{c^4}E^{(1)}_{\alpha\beta} + O(\frac{1}{c^6}) \tag{48}$$

$$T_{\alpha\beta} = c^2T^{(0)}_{\alpha\beta} + T^{(1)}_{\alpha\beta} + O(\frac{1}{c^2}) \tag{49}$$

and the field equations $E_{\alpha\beta} = -(8\pi G/c^4)T_{\alpha\beta}$ split in a set of equations which no more contain the constant c:

$$E^{(0)}_{\alpha\beta} = -8\pi G T^{(0)}_{\alpha\beta} \quad , \tag{50}$$

$$E^{(1)}_{\alpha\beta} = -8\pi G T^{(1)}_{\alpha\beta} \quad , \text{etc...} \tag{51}$$

One obtains:

$$T^{(0)}_{\alpha\beta} = \rho_M(r)\delta_{0\alpha}\delta_{0\beta} \quad , \tag{52}$$

$$T^{(1)}_{\alpha\beta} = \frac{1}{2}\rho_M(r)\left[\nu_0(R) + \nu_0(r) - \lambda_0(r) - 2\sigma_0(r)\right]\delta_{0\alpha}\delta_{0\beta} - a_{\alpha\beta}p_M(r) \tag{53}$$

where $a_{\alpha\beta}$ is the tensor of Minkowskian metric in spherical coordinates, and $p_M(r)$ is the equilibrium pressure:

$$p_M(r) = -\int_r^R \rho_M(r)\Phi'_M(r)dr \tag{54}$$

expressed by means of the Newtonian potential:

$$\triangle_r \Phi_M = -4\pi G \rho_M \tag{55}$$

The equations 50, written in explicit form, are the following:

$$\begin{aligned}\triangle_r \sigma_0 + \frac{1}{r}\left(\sigma_0' - \lambda_0'\right) + \frac{1}{r^2}\left(\sigma_0 - \lambda_0\right) &= -8\pi G \rho_M(r) \\ -\frac{1}{r}\left(\sigma_0' + \nu_0'\right) + \frac{1}{r^2}\left(\lambda_0 - \sigma_0\right) &= 0 \\ -\frac{1}{2}\triangle_r \sigma_0 - \frac{1}{2}\triangle_r \nu_0 + \frac{1}{2r}\left(\nu_0' + \lambda_0'\right) &= 0\end{aligned} \tag{56}$$

Assuming for σ_0 a value $\sigma_0 = 2\alpha\Phi_M$, one obtains the solution of the system 56 as:

$$\nu_0 = -2\Phi_M \quad , \qquad \lambda_0 = 2\alpha\Phi_M(r) + 2(\alpha-1)r\Phi_M' \quad , \qquad \sigma_0 = 2\alpha\Phi_M \tag{57}$$

The system 51 contains not only the quantities $\nu_1, \lambda_1, \sigma_1$, but the quantities $\nu_0, \lambda_0, \sigma_0$ as well. Inserting in 51 the values 57 of the zero-labelled quantities, one obtains 51 in explicit form, namely:

$$\begin{aligned}&\left[\triangle_r \sigma_1 + \tfrac{1}{r}\left(\sigma_1' - \lambda_1'\right) + \tfrac{1}{r^2}\left(\sigma_1 - \lambda_1\right)\right] + \left(\alpha^2 + 2\alpha - 2\right)\Phi_M'^2 = \\ &8\pi G\left\{(\alpha-1)\left[\Phi_M - (\alpha+1)r\Phi_M'\right]\rho_M + \left[\tfrac{GM_0}{R}\rho_M + p_M\right]\right\}\end{aligned} \tag{58}$$

$$\left[\frac{1}{r^2}\left(\lambda_1 - \sigma_1\right) - \frac{1}{r}\left(\sigma_1' + \nu_1'\right)\right] + \left(\alpha^2 - 2\alpha + 2\right)\Phi_M'^2 = -8\pi G \rho_M \quad , \tag{59}$$

$$\begin{aligned}&\left[-\tfrac{1}{2}\triangle_r \sigma_1 - \tfrac{1}{2}\triangle_r \nu_1 + \tfrac{1}{2r}\left(\nu_1' + \lambda_1'\right)\right] - \left(\alpha^2 - 2\alpha + 2\right)\Phi_M'^2 = \\ &8\pi G\left\{\tfrac{1}{2}\left(\alpha-1\right)^2 \rho_M \cdot r\Phi_M' - p_M\right\}\end{aligned} \tag{60}$$

We multiply 60 by 2 and thereafter add it with 59 and 58; one obtains:

$$\triangle_r \nu_1 + 4\left(1-\alpha\right)\Phi_M'^2 = -8\pi G\left\{\left(1-\alpha\right)\left[2r\Phi_M' - \Phi_M\right]\rho_M + \left[\frac{GM_0}{R}\rho_M - 2p_M\right]\right\} \tag{61}$$

Now, let us define a new function Φ by mean of the formula:

$$\Phi = \Phi_M + \frac{1}{c^2}\left[\left(\alpha-1\right)\Phi_M^2 - \frac{1}{2}\nu_1\right] \tag{62}$$

Accordingly,

$$1 - 2\frac{\Phi}{c^2} + 2\alpha\frac{\Phi^2}{c^4} = 1 + \frac{1}{c^2}\nu_0 + \frac{1}{c^4}\left(\frac{1}{2}\nu_0^2 + \nu_1\right) = e^{\nu} = g_{00} \quad . \tag{63}$$

From 61 and 62 we derive the Poisson type equation for Φ:

$$\triangle_r \Phi = -4\pi \frac{G}{c^2} \rho_E \quad , \tag{64}$$

$$\rho_E = \left(1 - \frac{GM_0}{c^2 R}\right) \rho_M c^2 + (\alpha - 1)\left(\Phi_M + 2r\Phi'_M\right) \rho_M + 2p_M \quad . \tag{65}$$